Coin flipping

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Bachelor Thesis

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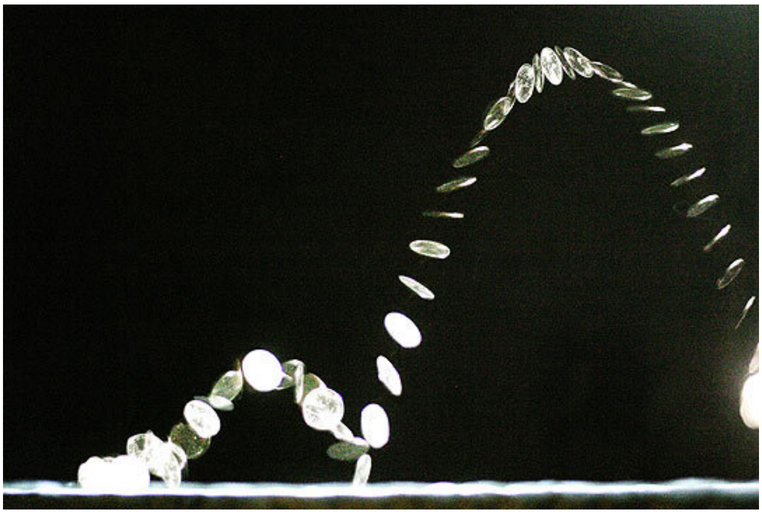

Figure 1: Is coin flipping really as random as it is considered to be?

Coin flipping is considered to be the ultimate random process.
In sports a coin toss decides which team starts the game, in every day life it's a decision-making-aid, maths regards coin flipping as the simpliest random experiment and some just see it as a gamble.
If you analyse the coin's motion physically, the flip is just a mechanical problem that should be - according to Newton's laws - well determined without any chance in the result.
We are confronted with a problem: If physics strictly determines heads or tails, why is the coin commonly used as a random device?
Answering that question is the concrete purpose of this thesis; a detailed illustration of the argumentation on this subject is given in section 2.7. The rest of the thesis is structured as follows: As there is a certain correlation between randomness and chaos, you will get a general idea of chaos theory in chapter 2. Chapter 3 analyses the coin flipping process physically and shows a model for predicting the result of a coin drop. Moreover you will get an overview of articles on this topic. The final part is dedicated to coin dropping experiments which verified the coin dropping model introduced in chapter 3 at low velocities.

2 Chaos theory

Even behind chaos there is science!
The chaos that chaos theory deals with is not a state as the commonly used term refers to but a form of behaviour of a system in the course of time, physically spoken its dynamics.
A chaotic system is in certain circumstances extremely sensitive to its initial conditions; that means that it is not possible to predict the system's longtime behaviour.

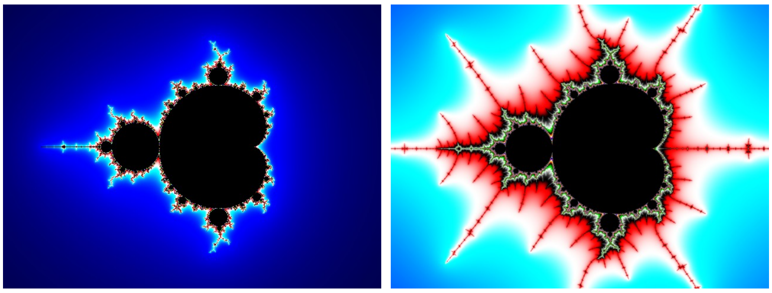

Figure 2: Fractals are the graphic depictions of chaotic systems. Fractals show self-similarity as you can see above; the right picture is just an enlargement of a region in the left one.

Chaos theory has been able to discover particular patterns behind this chaotic behaviour. Chaotic behaviour often occurs in amplified (e.g. through feedback) systems; friction can avoid chaotic motion.
An example for a technical implementation of chaotic systems is a random number generator, which is made sensitive to initial conditions by stretching and folding[1]. [5]
The mathematical description of chaos are nonlinear differential equations which can (in most cases) merely be solved numerically. So the development of chaos theory just started when computers were powerful enough to support the mathematical treatment of chaos; a lot of research was done from the 1960's to the 1980's. At that time scientists like Edward N. Lorenz, Benoit Mandelbrot and Mitchell Feigenbaum became famous for their studies on chaos theory.
But already at the end of the 19^{th} century Henri Poincaré contributed a lot to the developments made about a hundred years later, when he thought about the stability of orbits in our solar system.
In physics chaos has already been known since the 1920's, when the advances in quantum mechanics showed that determinism can't describe nature properly.
After the hype of chaos research in the middle of the 1990's the science had to meet with more and more criticism.
Some of these arguments are listed below:

- Chaotic effects predominantly occur after a long time but the models tend to get more inaccurate over a long period of time. Moreover, you have to deal with more and more information (converging into an infinite amount) to prognosticate the behaviour; that leads to more approximate models. Do these models still simulate chaos accurately?
- In physics you usually test and prove a theory with experiments, but adapting chaotic models and experiments is often not possible.
- It is difficult to see chaos theory as a comprehensive science because of the many

[1] This stretching and folding procedure is also known as Baker's map. The name derives from a repeating kneading process for dough: the dough is cut in half, the two pieces are put one upon the other and compressed and so on. Raisins in the dough will so be randomly distributed. [6], [7], [8]

interdisciplinary aspects of the theory (physics, mathematics, meteorology, economics, ...). So the whole theory resolved into areas which concentrate on the certain branches of interest.

Still there are many definitions which all chaos fields have in common; these will be explained in the following sections. [1] - [5]

2.1 Deterministic chaos

Chaos, as we are talking about here, is scientifically called *deterministic chaos*. This term seems to be an oxymoron; however, it describes the fact that the laws of nature are valid but that the development of the system is not predictable. Furthermore the system is extremely sensitive to the initial conditions and although there is an infinite variety of states, the system will only assume certain states.
So the principle of linearity (equal causes have equal effects [11]) is violated - small changes in the initial conditions lead to totally different outcomes!
This principle, which is something we regard as natural[2], is thus not the right approach to chaos. The changes in initial conditions grow rather exponentially in time.
[1] - [4], [9] - [12]

2.2 Predictability

Theoretically, it's possible to calculate the future behaviour of a chaotic system; you just need to know the initial conditions precisely enough - but so precisely that there is no measurement that can guarantee this precision. With our instruments we can only make predictions for very small time intervals; the length of these intervals can be characterized by the *Lyapunov exponent*.
Another limit, actually the lower bound of predictability, is quantum mechanics and Heisenberg's uncertainty principle. [1], [2], [13]

2.3 Phase space

Phase space gives a clear graphical overview of the dynamics of a system. The state of the system at every single moment is depicted by one point in phase space. Every degree of freedom or parameter of the system is represented by an axis of a multidimensional space. So the dynamics of the system are described by a curve in the phase space. For a harmonic oscillator the phase curve is an ellipse in a two-dimensional phase space which is spanned by the position and the oscillator's velocity or kinetic energy. A periodic motion corresponds to a closed curve around the point of origin. [1], [2], [14], [15]

2.4 Attractors

In certain circumstances systems with different initial conditions converge to the same behaviours. This can be a point or curve in phase space that is called attractor; it describes the final state of a dynamical system[3]. Special forms of attractors, so-called strange

[2]If you hit a ball twice as hard, it will fly away twice as quickly! [11]
[3]A dynamical system is the mathematical model of a time-dependent process. [16]

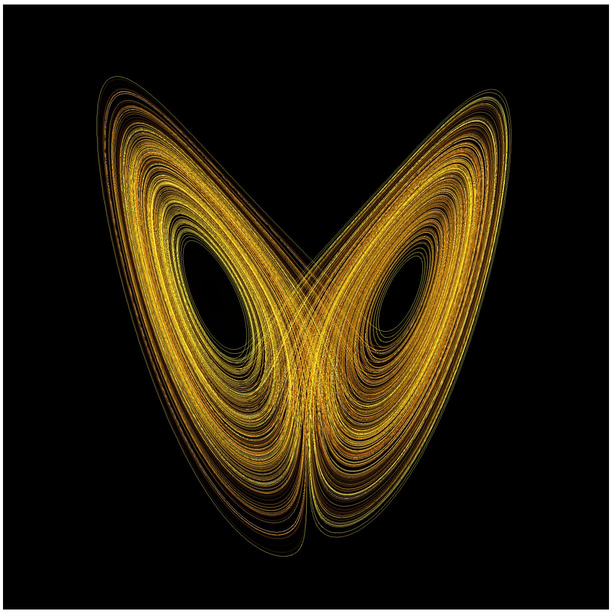

Figure 3: A plot of the Lorenz attractor for values $\sigma = 10$, $\beta = \frac{8}{3}$, $\rho = 10$ ($\rightarrow$ eq. 1, 2, 3)

attractors, occur in chaotic systems. Their dimensions are fraction numbers and their structures are fractal[4]. Strange attractors can only arise in systems that have three or more dimensions. One of the most famous strange attractors is the *Lorenz attractor*.
The equations that govern the Lorenz attractor are:

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

where σ is called the Prandtl number and ρ is called the Rayleigh number. All σ, ρ, $\beta > 0$, usually $\sigma = 10$, $\beta = \frac{8}{3}$ and ρ is varied. [22]
The equations are based on the Navier-Stokes equations that describe the motion of fluids. This attractor and the non-linear equations (1), (2), (3) were introduced by Edward Lorenz in 1963 when he wanted to find a model for the states of the earth's atmosphere for a longtime weather forecast.
Closely linked to this strange attractor is the *Butterfly effect*. [1], [2], [17] - [26]

2.5 Butterfly effect

Butterfly effect is the more popular term for the concept of sensitive dependence on initial conditions in chaos theory.

[4]Fractal means that the pattern shows self-similarity, thus the whole has the same shape as one or more of the parts ($\rightarrow$ figure 2). Its geometrical structure can't be described by Euclidean geometry.

Figure 4: A coin flip as we perform it in every day life is actually the toss of a two-sided die. There is even a connection to toast that lands butter side down when accidentally falling off a table and to the falling cat theorem. This is the scientific treatment of the fact that a cat is always able to land on its feet, no matter which way up it started. [30], [31]

It was formed by the meteorologist Edward Lorenz, who did several computer calculations on weather forecasts in 1963. He accidentally came across the chaotic behaviour of his model when he used rounded data to save calculating time.
It's not clear if the effect's notation comes from Edward Lorenz' citation *"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"* or from the form of the strange attractor that is linked to this effect. [1], [2], [18], [27], [28], [29]

2.6 Examples for chaotic systems

Many activities in nature are based on nonlinear processes; so they are described by chaotic models. For instance:

- The *weather* is one of the most famous chaotic systems. We are (with our measuring instruments) not able to predict it exactly, we don't know all the initial conditions. Even in the future the predictability will not improve because the amount of information needed for precise forecasting is too big.
- The *double pendulum* is a popular simple experiment to show chaotic behaviour.
- The *heart rhythm* can be seen as chaotic signal. Scientists try to predict sudden cardiac death with the help of chaos theory but there are many critics of that 'chaotic medical approach'.
- *Turbulent flows* are also known to be chaotic. This aspect is an interesting point for coin flipping, which is treated in section 5.2.

[1], [2]

2.7 Chaos and randomness in coin flipping

We will try to answer the question why coin flipping is considered to be random and under which conditions chaos occurs in coin flipping.

There is a significant difference between the mathematical and physical meaning of randomness and chaos. Mathematically and commonly spoken randomness[5] describes the unpredictability of the outcome whereas physics in fact sees randomness as an inaccuracy in the adjustment of initial conditions and regards chaos to be the sensitivity to initial conditions.
Thus you could say that *not the outcome is random but the initial conditions are.* This imprecision in initial conditions doesn't lead to chaotic behaviour because linked to a certain adjustment of initial conditions there is a physically determined outcome. This can in certain 'areas of initial conditions', more precisely in certain basins of attraction[6], be very sensitive to small variations. You find these basins of attractions in those sectors where there is change between heads and tails (change between black and yellow areas in figure 5). In these regions the outcomes may seem chaotic but in fact you have to expect uncertainty in the results. As long as you keep your method of coin flipping simple (as in the experiments described in chapter 5) you can't call coin flipping chaotic; there is no deterministic chaos in coin flipping.
If you introduce more complex models of coin flipping, that depict the coin flipping process more realistically, the amount of information needed to determine the outcome of the flip can become too big. Coin flipping can then be compared to other chaotic examples (→ section 2.6) and regarded to be chaotic.

So you should say that the initial conditions are chosen randomly, which means that you can't adjust or measure the initial conditions exactly enough to know the well determined outcome (see Vulović' definition of 'random' outcomes in 4.1).
In other words: It's clear that the coin thrower brings the randomness into the flip.
Identifying a coin flip as a randomizer is legitimate as long as the (human) coin flipper doesn't concentrate on setting the initial conditions so that a known final outcome can be expected. When you perform a coin flip you usually don't know all the initial conditions that determine the outcome of the flip. Actually you don't think and shouldn't think about it, because then the coin flip wouldn't be random anymore. Moreover, most people aren't aware of the determinism that lies in coin flipping, so it's just natural to think that the result is random. [32]

3 Newton's laws

The physical description of the coin flipping motion are, as for any other mechanical process, the Newtonian equations of motion.
By setting up a rather simple model and by solving the corresponding differential equations you are able to determine the result of the flip.
As the experimental part of this thesis concentrates on coin dropping it is possible to use a very simplified mathematical model, where only a one-dimensional motion has to be analysed and just a few parameters have to be known to predict the coin drop's outcome.

[5] in this respect often called chaos as well
[6] A basin of attraction is the set of initial conditions leading to long-time behaviour that approaches a certain attractor (here either heads or tails). [33]

Moreover it neglects air resistance and bouncing and the coin behaves like a point mass. The coin's motion consists of a falling (= constantly accelerated) and a rotating (= uniform motion) part, which can be described individually.
The falling height determines the time that the coin has for its rotating motion.

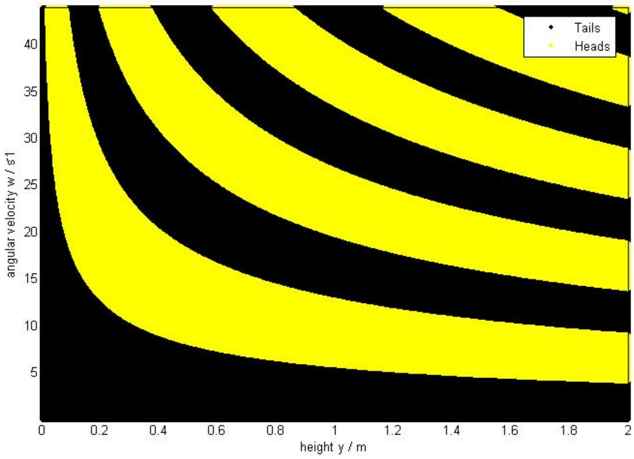

Figure 5: The plot shows the theoretically expected results of a coin drop (starting angle of the coin 30°, tails up). By choosing a certain angular velocity you can read out the corresponding result to a certain height (up to 2 m) or vice versa.

So first of all you have to solve the differential equation

$$\ddot{y} = g \tag{4}$$

with y the dropping height and g the gravitational acceleration. The well-known solution is

$$y = \frac{1}{2}gt^2 + v_y t + y_0 \tag{5}$$

where v_y is the starting velocity and y_0 the starting height, which are in this approach both to be set to *0*. With this configurations the coin touches the ground after the time

$$t = \sqrt{\frac{2y}{g}} \tag{6}$$

Now you can determine the angle ϕ at which the coin will hit the ground by solving

$$\ddot{\phi} = 0 \tag{7}$$

So the final result will be

$$\phi = \omega t \tag{8}$$

where ω is the angular velocity of the coin.
By defining either heads or tails as the starting configuration and applying the modulo

operation on the angle ϕ, you are able to determine the landing angle of the coin and can so tell whether the coin will show heads or tails after dropping from a certain height.

If you plot[7] the results depending on variable dropping heights and angular velocities of the coin you can see the changes between heads and tails as in figure 5.

To test if this model is really able to predict the result of a coin drop a coin dropping apparatus was built that reproducibly drops 1-€-coins. The detailed setup of the device and the results of these experiments will be discussed in chapter 5.

4 Articles on coin flipping

Since the 1980's there have been several papers on coin flipping; above all the scholarly pieces of Vulović & Grange and in 2007 the one of Diaconis and his team. They both refer to Joe Keller's analysis who was the first that approached the topic scientifically.
If you search the internet you find a lot of popular science articles on the subject. They predominantly focus on Diaconis' work. A couple of articles deal with biased coins in coin spinning; the U.S. penny and the German and Belgian 1-€-coins are examples for coins with uneven mass distributions that lead to biased results.
The next sections will concentrate on Vulović' and Diaconis' papers. [32], [34] - [40]

4.1 Randomness of a true coin toss - Vladimir Vulović and Richard Prange

Vladimir Vulović and Richard Prange from the University of Maryland published their paper on the randomness of a coin toss in 1986. They created and numerically solved a physical model of a coin flip, which can in general also be applied to other randomizing devices. They conclude that coin tossing is not random.
Their 'true' coin describes a totally symmetric and infinitely thin[8] coin, that obeys Newton's equations of motion. Fluctuations of air (the coin is falling in vacuum) and thermodynamic and/or quantum fluctuations of the coin and the surface it falls on are neglected. Furthermore the coin has only one axis of rotation, so it can't wobble.
The initial conditions that are mentioned are the position, configuration, momentum and angular momentum of the coin just after it leaves the flipper (human or device). Vulović and Prange say that the initial conditions can be set to a certain accuracy they call ϵ, which is generally not precisely determined.
They use the model of Joe Keller. This simplified model does not allow the coin to bounce and it determines heads or tails by the angle the coin first touches the landing surface.
Vulović and Prange plot the outcomes depending on the coin's potential energy and its rotational energy (→ figure 6).

In figure 6 you can see that a sequence of coin tosses will deliver random outcomes if the uncertainty ϵ is large compared to the width W of the stripes, which characterize the

[7] e.g. with MATLAB
[8] landing on the coin's edge is not possible

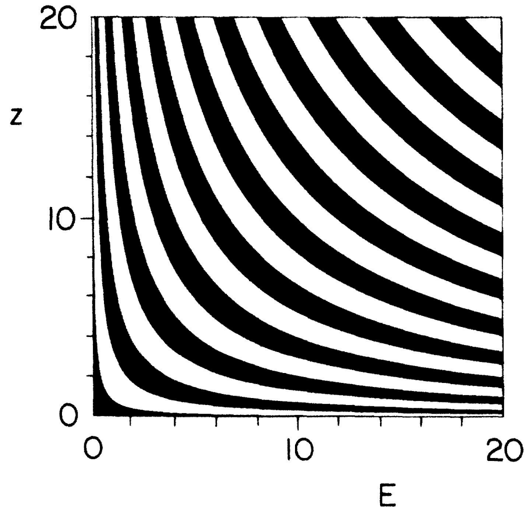

Figure 6: Outcome of the simplified model of Joe Keller as a function of the initial excess height z and the initial rotational energy E. Dark regions correspond to heads. The plot shown in figure 5 is proportional as the parameters there are the dropping height and the coin's angular velocity. [32]

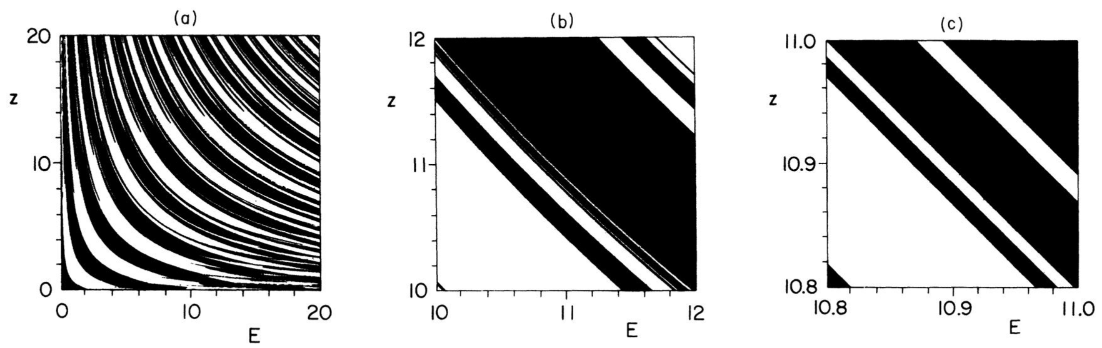

Figure 7: (a) Same as in figure 6 but for a more realistic model, including bouncing of the coin from the surface. (b) The enlargement of (a) reveals additional structure. (c) The enlargement of (b) shows smooth boundaries; no further structure can be resolved. [32]

basins of attractions.
In their opinion that is the basic reason for coin tossing to be considered random. This actually means that *the coin thrower is the one who brings the randomness into a coin flip.*
So the condition for the outcome to be random is $W << \epsilon$. Note, that ϵ depends on the flipping mechanism whereas W is intrinsic to the coin. If you decrease ϵ continuously you can reach the condition that $W >> \epsilon$, which characterizes deterministic devices.
Vulović and Prange introduce more realistic and thus non-linear models (→ figure 7) where the coin is allowed to bounce on the landing surface. They add that even more realistic models have even more complicated basins of attraction, the stripes' structure is more complex.

As the bouncing model plot shows, the areas of clear outcomes are sometimes quite narrow and so all the (initial) parameters have to be set quite accurately to ensure the desired result. To some extent you can say that bouncing enhances randomness. [32], [34]

4.2 Dynamical Bias in the coin toss - Persi Diaconis

In 2007 the mathematician Persi Diaconis and his team from Stanford University published a detailed paper on coin flipping.
Their approach is quite mathematical but they also built a coin flipping device to experimentally demonstrate the reproducibility and non-random behaviour of coins and did a series of hand flip experiments to test their assumptions.
Diaconis' group concentrates on natural flipped coins which are caught in the hand. They neglect air resistance but take precession into account, which means that the direction of the axis of rotation changes as the coin flips.
They show that vigorously flipped coins tend to come up the same way they started (the probability is about 0,51). In their opinion there is bias in coin flipping, determined by the coin's initial conditions, especially by one single parameter, the angle between the normal to the coin and the angular momentum vector. Measurements of this parameter were done with the help of high-speed photography.
Diaconis also gives an overview of studies that had already been done on coin flipping, coin spinning and other random devices. Again, Poincaré was one of the pioneers with his analysis of roulette in the late 19^{th} century. The first analyser of the coin flipping phenomenon was Joe Keller in the 1980's, whose mathematical approach is similar to that shown in chapter 3; the number of revolutions of the coin determines the outcome.
Keller also introduced a 'total cheat coin', a coin that always comes up the same way it started. The coin is hit exactly in center, so there is no turning, just spinning. That means that the angular momentum vector lies along the normal of coin, so there is no precession.
Another special flip is often performed by magicians. They are able to let their coin flip look like a fair one (which would include turning of the coin); in fact they carry out a flip where the angle between the angular momentum vector and the normal of the coin is less than 45°, so the coin may wobble but never turns over (precession occurs here!).
Furthermore, Diaconis takes a look at papers on coin spinning. Coin spinning on the edge seems to be, according to the fact that the weight is not uniformly distributed over the coin (due to embossing), even more biased than coin flipping. He adds that this inhomo-

geneity in mass distribution does not affect the coin flip as some papers explain and even Diaconis' experiments with coins made with lead on one side and balsa wood on the other showed no bias.
In coin spinning the outcome depends on the shape of the coin's edge and the exact center of gravity. Again Diaconis, who used to be a magician, emphasizes that magicians use manipulated coins (their edges are slightly shaved) to control the outcome.
Some American scientists enter the question "How thick must a coin be to have a probability of $\frac{1}{3}$ of landing on the edge?", others concluded that a U.S. nickel will land on its edge about 1 in 6000 tosses. This possible outcome is neglected in Diaconis' approach (as well as in this one here).

A legitimate question is whether tossed coins landing on the floor are less fair than those caught in the hand. Diaconis considers this to be true, whereas most people think that coins landing on the floor are more fair. The fact is, that a coin that lands on the floor may spin around on its edge before it finally comes to a rest and thus is biased. However, you should also bear in mind that the coin will bounce and that is, as mentioned in section 4.1, a very non-linear process. So the question can't be answered satisfactorily.
For Diaconis and his team it is also important to define the exact angular velocity of the flipping coin, so they find two different ways to measure it. The first one is a low-tech experiment, where there is a ribbon attached to the coin. The number of flips is determined by unwinding the ribbon until it is untwisted. They checked that the attached ribbon didn't affect the flight of the coin notably. For the more high-tech way to measure the number of flips they used a high-speed slow motion camera.
At the end of the paper Diaconis points out assumptions they made in their approach. In their coin flips[9] they 'avoid' randomness to some extent. They always start their flips with heads up, a method that will not be used in 'real life coin tossing'. So their coin flips are a bit more determined.
In addition Diaconis underlines that he thinks that variations in the 'catching height' do not really affect their conclusion of biased coin flips. Tossed coins (starting heads up) will spend more time of their total flight time heads up, independently from the time of flight and catching height respectively.
Diaconis' conclusion:

> *"Despite these important caveats we consider the bias we have found fascinating. The discussion also highlights the true difficulty of carefully studying random phenomena. If we have this much trouble analyzing a common coin toss, the reader can imagine the difficulty we have with interpreting typical stochastic assumptions in an econometric analysis."*

They also consider Keller's analysis to be a good approximation for tossed coins and are convinced that coin flipping is under specified conditions strictly determined and thus not random. [34] - [42]

[9]Their human coin flips were caught without bouncing at approximately the same height.

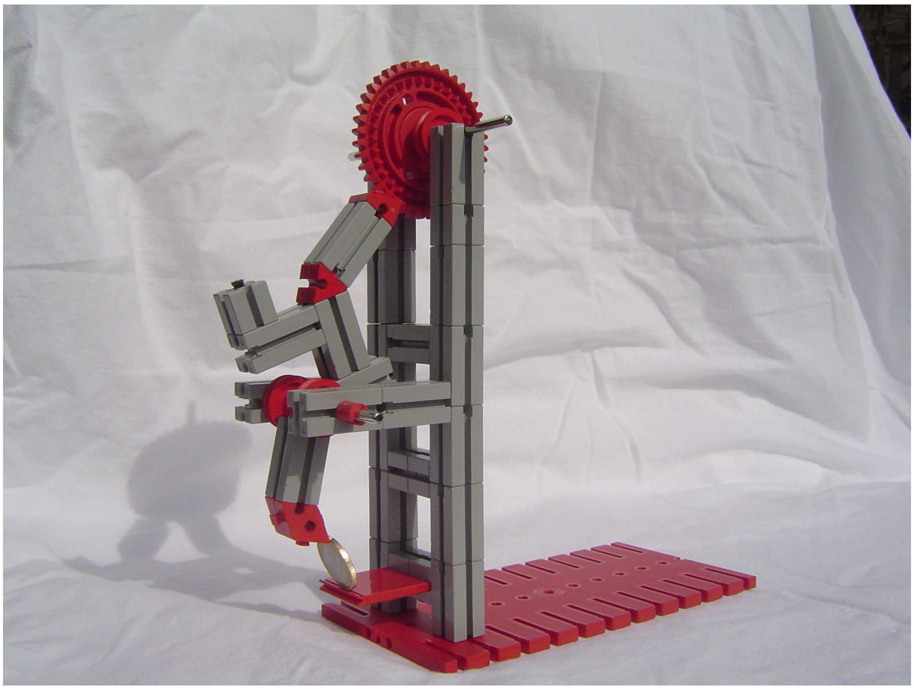

Figure 8: Coin dropping apparatus realized with *fischertechnik*

5 Experiments

To guarantee a simple mathematical model for the coin's motion as introduced in chapter 3, also the experimental approach to the subject was focused on coin dropping. The intention was to test the model with simple methods and see if it was possible to show the assumed determinism in coin dropping. Therefore a way of dropping coins reproducibly had to be found. Here, coin dropping makes it a lot easier to fulfill the requirements of always having the same (initial) conditions, because the only actual parameter that influences the drop's result is the falling height.

The coin dropping device was finally realized with *fischertechnik*, a construction toy.
The two most important elements in the dropping process are the release mechanism and the landing surface. Both factors should contribute as few (initial) conditions as possible. It is essential that the coin is just clamped in its starting position, so that the release mechanism doesn't affect the coin's movement notably (especially gives the coin only little initial velocity so that the mathematical model describes the process correctly).
The landing surface has to ensure that the result the coin shows when it first touches the surface is also the final outcome (no bouncing, spinning or rolling). Therefore finely granulated sand (sandbox sand) was used that was put in a box so that the whole coin dropping apparatus was portable. (→ figure 8 - 13)

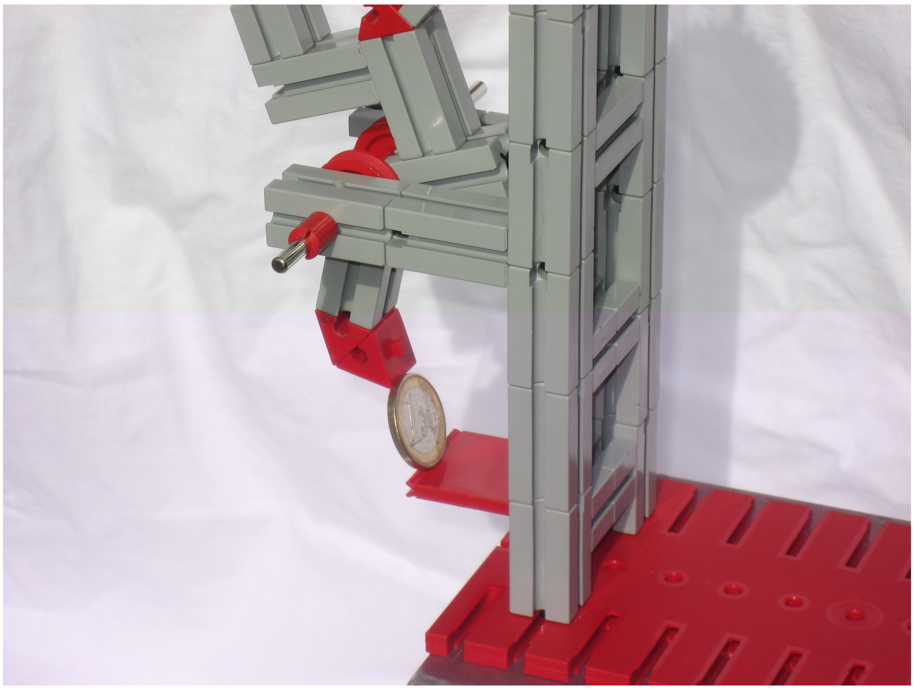

Figure 9: Starting configuration of the coin

Figure 10: Apparatus before and after a coin drop

Figure 11: Coin drops

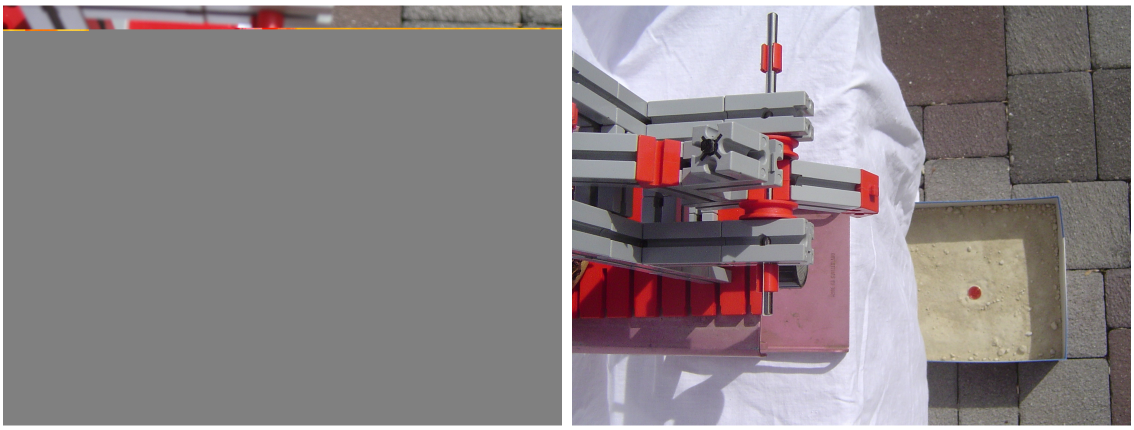

Figure 12: Coin drop and its result

Figure 13: Higher coin drop setup

Figure 14: The test candidates, three 1-€-coins and the landing surface, finely granulated sand box sand

5.1 Implementation and discussion of results

The experiments were conducted at low velocities and at heights from approximately 7 cm up to 1,70 m; the test objects were several 1-€-Coins.
First of all the apparatus was put on a completely extended lifting platform, which was used for fine height adjustment. It was examined if it was possible to reproducibly drop coins from that height (approx. 25,8 cm). This test series was completed successfully. Even repeating coin flips from the same height after removing and rebuilding the device didn't influence the results.

The experiments were based on the plot of the theoretical results (→ figure 5); as it is quite difficult to determine the angular velocity just the height was varied in the experiments and the coin drop's results corresponding to a certain height were noted.
In comparing these results to the plot and to values from rough calculations[10], an estimated value for the angular velocity was found, that is marked in figure 15.

Then the attention was drawn a bit more on reproducibility; if reproducibility was found, it would be possible to act on the assumption that coin dropping at low velocities and at heights up to 1,70 m was *not* random (→ figure 16).
A couple of coin drops in regions where the results should be - according to the model - well-determined, were done.
Again it was possible to find accord with the theoretical expectations. So the initial conditions of the apparatus and the conditions added by the landing surface were adjusted precisely enough to ensure same outcomes for a coin dropped from the same height.

As the experiments were concentrated on the intervals of change between heads and tails,

[10] by applying Newton's laws on observations of the number of rotations of the coin at 5 cm dropping height

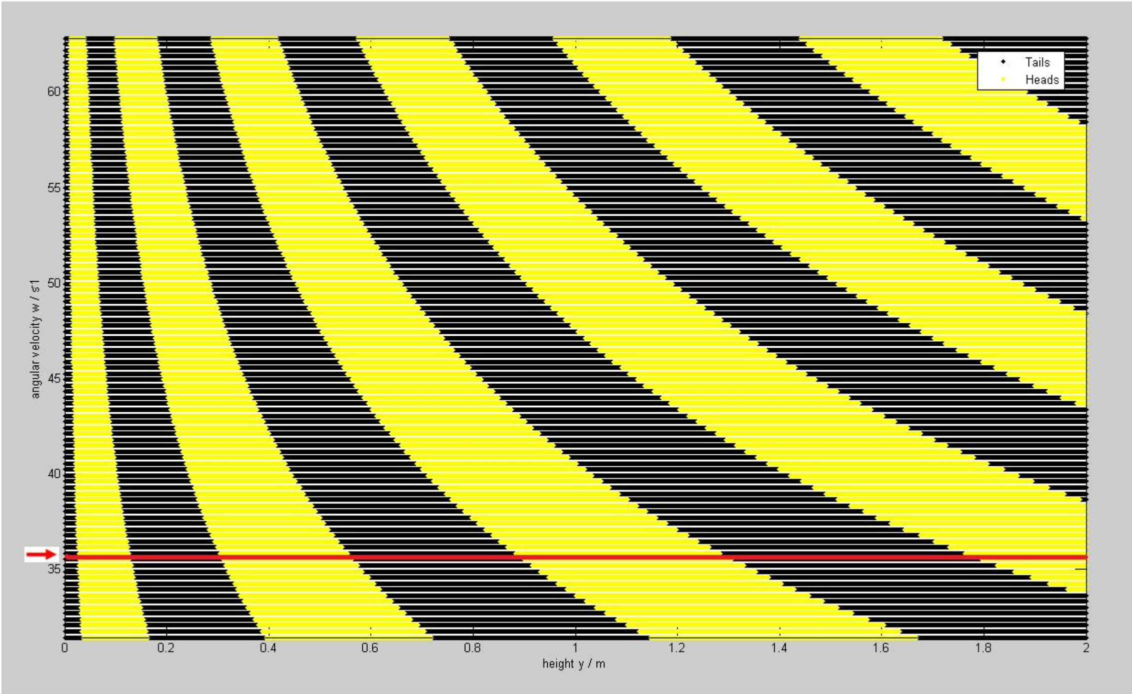

Figure 15: The estimated value of angular velocity of the dropped coin is marked. By varying the dropping height the results were verified along this line of same angular velocity.

random behaviour could be observed, as the results became quite sensitive to the initial conditions and the landing surface. In these regions the apparatus can not be adjusted accurately enough to guarantee just millimeters of change in height and the starting angle. But predominantly this randomness derives from the unevenness of the landing surface that leads to ambiguous results, as can be seen in figure 18.
In figure 17 you can see that the region of transition stretches out over about 2 cm.

When the coin was dropped from heights higher than 1,70 m bouncing falsified the result more and more because the coin got quite fast and even the finely granulated sand couldn't stop it accurately. But for real coin drops/flips these heights are not really relevant.

In summary the experiments showed:

- very good reproducibility in heights where the result is exactly determined and
- expected uncertainty in those regions, where there is change between heads & tails.

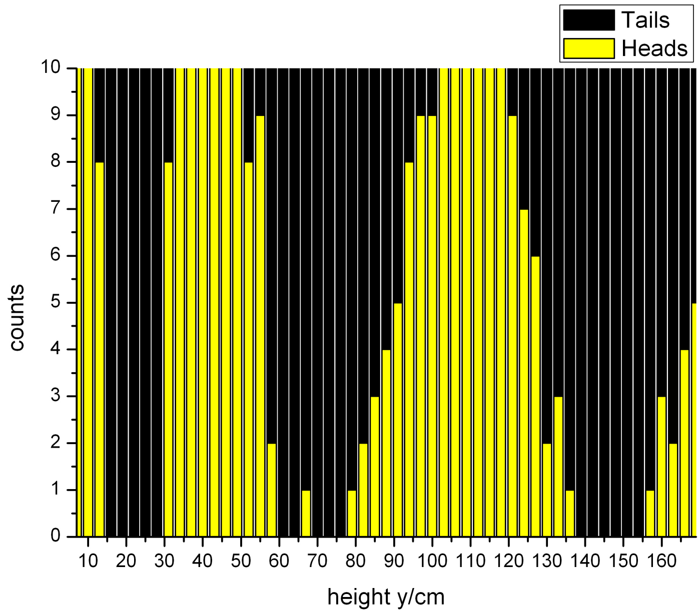

Figure 16: Plot of the experimental data - ten drops every 3 cm (starting with 7 cm) up to a height of 1,70 m.

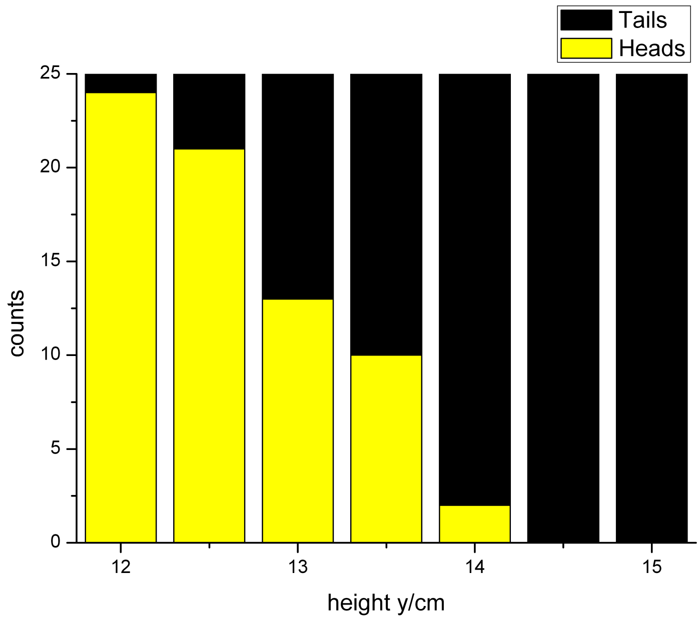

Figure 17: Plot of the experimental data focusing on a region of change between heads and tails.

Figure 18: A result that can be observed at dropping heights within the region of transition between heads and tails.

5.2 Further considerations

The experiments described above focused on low velocities and low heights and neglected air resistance. For further considerations about coin flipping air resistance and its effects should be discussed here.

Coin dropping allows the coin just to spin rather slowly. The flow around the coin doesn't affect the coin notably. But if you flip it vigorously the flow will be turbulent and can cause chaotic behaviour that can't be described by the equations mentioned in chapter 3. So flipping the coin hard is one possibility to make the outcome unforeseeable.

Another way of bringing chaos into the coin flipping process is to drop it from bigger heights or by performing a high flip. Then air resistance will definitely affect flipped and dropped coins (up to 1,70 m these effects were not observed) [34].

Especially for coin dropping the landing plays a vital role; bouncing on the landing surface can cause unpredictable results (as observed for heights $>$ 1,70 m, as even finely granulated sand can't stop the coin adequately). The mathematical depiction of such drops is quite complex and can lead to chaotic behaviour as the basins of attraction can show fractal structure [32].

The experiments in chapter 5 were not affected by the factors mentioned. Thus, this way of coin dropping can be considered to be predictable.

6 Conclusion

The purpose of this thesis was to examine where the randomness in coin flipping comes from.
Working through published studies on coin flipping, setting up a mathematical model, constructing a coin dropping apparatus and implementing a series of coin dropping experiments helped finding an answer as well as dealing with chaos theory.

The coin flipping process is a problem of classical mechanics. So the outcome of a flip is well-determined by Newton's laws when the flip is performed under the same conditions. Carrying out similar flips leads to reproducible results. This statement was successfully tested with the coin dropping apparatus constructed. Here for this purpose dropping the coin was a way to keep the amount of initial conditions down to a minimum.
As long as the coin's motion can be described with the simple Newtonian equations, the coin flip doesn't show any chaotic behaviour; the coin flip is not extremely sensitive to the initial conditions. There is no deterministic chaos.

Randomness appears in the coin flip when the adjustment of the initial conditions is not accurate enough to ensure a certain outcome. This especially occurs in regions where there is change between heads and tails. Still by increasing the accuracy of having the same initial conditions you are able to obtain reproducible results.
Of course, there is a limit to this accurateness. Particularly in performing a natural coin flip, e.g. when deciding which team starts a football game, the coin thrower doesn't (shouldn't) concentrate on this aspect.
So if you train your coin flips, you will be able to get nearly always the same outcomes; using a coin flip as a random device in such circumstances is, of course, neither random nor fair.

7 Appendix

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